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A theory of transversely isotropic fluids

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The present paper proposes a theory for the mechanical behaviour of a fluid with a rigid microstructure. The microstructure is described by a director frame of three vectors and a second-order tensor $\mathbf{W}$ and its gradient are proposed as measures of the kinematics of this frame. When the frame is spinning without deforming, $\mathbf{W}$ reduces to the director spin velocity. Postulating the existence of a couple stress in addition to the classical Cauchy stress, the linear constitutive equations for such a structured fluid are derived and then specialized to the case of transverse isotropy.

These equations are used to study rectilinear shearing flow. When $\nabla\mathbf{W} = 0$, the condition for a non-interacting substructure, the results of the theory are shown to be in agreement with the work of Jeffery and of Ericksen. For mutually interacting substructure particles, $\nabla\mathbf{W} \neq 0$, a geometric analysis of the non-linear differential equations is performed in order to exhibit the effects of particle concentration on the flow kinematics.

1. Introduction

'In both physical and biological science, we are often concerned with the properties of a fluid, or plasma, in which small particles or corpuscles are suspended and carried about by the motion of the fluid.' This statement by G. B. Jeffery (1922) introduces his analysis of the modification produced in the motion of a viscous fluid by the presence of a single ellipsoidal particle. This analysis of the influence of a single particle can describe at best a dilute suspension since particle interactions are ignored. Oseen (1933) reviewed the work of Anzelius (1931), who attempted to formulate the laws of motion for a liquid with oblong molecules. Anzelius postulated expressions for the dependence of stress and moment at each point of the fluid as a function of the classical kinematic rate of strain variable and of the orientation of the substructure particles.

Ericksen (1960*a*, *b*) formalized Anzelius's work and put it on a more rigorous foundation. Hand (1962) extended Erickson's ideas to account for spherical particles which may deform into ellipsoids. Ericksen (1960*b*) points out that his equations governing the motion of a particle with a single preferred direction can be shown to be the same as the equations obtained by Jeffery (1922) for the motion of an oblate or prolate spheroid. The preferred direction concides with the axis of revolution of the ellipsoid. However, Ericksen's equations do not account for rotation about this axis. Since in Jeffery's work the particles are

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treated as being infinitely far apart, his equations may be considered to be valid for a dilute or weak suspension of particles. Equivalently they may be regarded as those of a medium with non-interacting particles since the particles are so distant. By this convention, Ericksen's equations may be classified as those governing a fluid with non-interacting substructure.

Consider now the question of a non-dilute or concentrated suspension, i.e. one in which we may no longer ignore particle interactions. In a continuum theory, we regard each element of a material as possessing not only mass and velocity but also a substructure with which we associate a moment of inertia and a spin velocity. Then, at each point of the continuum, we must have not only a force per unit area (Cauchy stress) but also a double force per unit area (Mindlin stress) which will exert twisting and bending moments on elemental volumes.

Similar ideas have been used in statistical treatments which complement the continuum approach, as in the work of Grad (1952), Dahler & Scriven (1963), and Dahler (1965). In the above references, the idea of structured continua is developed in which couple stress (arising from double force with moment) is interpreted as a flux of spin momentum just as the Cauchy stress is interpreted as a flux of linear momentum.

Recently Eringen & Suhubi (1964) and Eringen (1964) have developed theories for 'simple micro-elastic solids' and 'simple microfluids', which take into account rigid rotation and deformation of the microstructure. These theories are an extension of the work of Mindlin (1964) which treated infinitesimal elastic deformations and introduced the general double stress tensor. The microfluid theory of Eringen seems to ignore the fact that a microstructure which is non-spherical or not randomly oriented will give rise to anisotropic effects.

In this paper we develop a continuum theory which will describe the motion of a fluid having a rigid microstructure. The kinematics of such a structured continuum are treated and the constitutive equations for Cauchy and couple stress are formulated in terms of the kinematic variables and microstructure orientation. The proposed theory is valid for a fluid with interacting substructure. The theory is then specialized to the case of rectilinear shearing motion of an incompressible structured fluid with a single preferred direction and the complete equations recorded. These equations are first solved for the case of negligible particle interaction in order to obtain a comparison with the results of previous theories. Finally, the equations are applied to examine the behaviour of a structured fluid with non-negligible particle interaction.

2. Kinematics of structured continua

In any treatment of a continuum with microstructure we must consider, to borrow the terminology of Dahler & Scriven (1963), both strain of position and strain of orientation.

2.1. *Strain of position*

Positional strain for a fluid is well known and will not be considered in detail. We simply recall that for a fluid one follows a line element $d\mathbf{x}$ as it moves with the fluid and observes the changes that take place. We take the convective derivative of the line element and find that the velocity gradients are a measure of the rate of deformation

$$d/dt\,(dx^i) = v^i_{;j}\,dx^j, \tag{2.1}$$

where the semicolon denotes the covariant derivative and only diagonally repeated indices are summed.

2.2. *Strain of orientation*

The concept of orientation strain in a solid was first considered by the Cosserats (1909) and later put on a rigorous basis by Ericksen & Truesdell (1958). We introduce a set of three vectors $\mathbf{d}_\alpha$ ($\alpha = 1, 2, 3$) at each point $\mathbf{x}$ of the fluid. These vectors may have a broad physical interpretation but we use them to represent the microstructure at each point of the fluid. Let $\mathbf{d}^\alpha$ be the reciprocals to $\mathbf{d}_\alpha$ such that

$$d^k_\alpha d^\beta_k = \delta^\beta_\alpha; \quad d^k_\alpha d^\alpha_m = \delta^k_m. \tag{2.2}$$

We can define a generalized magnitude by

$$D_{\alpha\beta} = \mathbf{d}_\alpha . \mathbf{d}_\beta = D_{\beta\alpha} = (\mathbf{g}_k d^k_\alpha).(\mathbf{g}_m d^m_\beta) = g_{km} d^k_\alpha d^m_\beta, \tag{2.3}$$

where $\mathbf{g}_k$ are the base vectors of the general curvilinear co-ordinates defined in the instantaneous configuration of the continuum and where $g_{km} = \mathbf{g}_k \cdot \mathbf{g}_m$ is the metric of the co-ordinate system. The quantities $D_{\alpha\beta}$ describe instantaneously the magnitudes and angles between the triad of vectors $\mathbf{d}_\alpha$. If $D_{\alpha\beta} = 0$ for $\alpha \neq \beta$, the triad is orthogonal; if $D_{\alpha\beta} = A\delta_{\alpha\beta}$ the triad is orthogonal and the vectors are of equal length.

We now consider the instantaneous rate of change of this generalized magnitude while moving with the fluid particle to which the triad is attached. From (2.3)

$$d/dt\,(D_{\alpha\beta}) = \dot{D}_{\alpha\beta} = g_{km}(\dot{d}^m_\beta d^k_\alpha + \dot{d}^k_\alpha d^m_\beta). \tag{2.4}$$

If we multiply equation (2.4) by $d^\alpha_p d^\beta_n$, we obtain on using equation (2.2)

$$d^\alpha_n \dot{d}_{\alpha p} + d^\alpha_p \dot{d}_{\alpha n} = \dot{D}_{\alpha\beta} d^\alpha_p d^\beta_n. \tag{2.5}$$

The solution of equation (2.5) for $\dot{D}_{\alpha\beta}$ is

$$\dot{D}_{\alpha\beta} = 2W_{(mn)} d^m_\alpha d^n_\beta, \tag{2.6}$$

where

$$W_{mn} = d^\alpha_m \dot{d}_{\alpha n} \quad \text{and} \quad W_{(mn)} = \tfrac{1}{2}(W_{mn} + W_{nm}). \tag{2.7}$$

Now, from equation (2.3), the condition that the director frame does not deform is given by $\dot{D}_{\alpha\beta} = 0$. From equation (2.6), it follows that the condition

$$W_{(ij)} = 0 \tag{2.8}$$

is a necessary and sufficient requirement for the directors to be rigid. This condition, however, in no way limits rotations of the triad. For a rigid triad $\dot{\mathbf{d}}_\alpha = \mathbf{\Omega} \times \mathbf{d}_\alpha$, where $\mathbf{\Omega}$ is the angular velocity of the frame. By using equation (2.2), and resolving $\mathbf{\Omega}$ into its components along the director frame, it follows that $\mathbf{d}^\alpha \times \dot{\mathbf{d}}_\alpha = 2\mathbf{\Omega}$, or equivalently,

$$2\Omega^i = \epsilon^{ijk} d^\alpha_j \dot{d}_{\alpha k} = \epsilon^{ijk} d^\alpha_{[j} \dot{d}_{\alpha k]}, \tag{2.9}$$

where $\boldsymbol{\epsilon}$ is the absolute alternating tensor and the square brackets about two indices indicate antisymmetrization with respect to those indices. From equations (2.7) and (2.8), we obtain

$$\Omega^i = \tfrac{1}{2}\epsilon^{ijk} W_{[jk]}, \quad W_{[jk]} = \epsilon_{ijk}\Omega^i = \Omega_{jk}. \tag{2.10}$$

We have, therefore, the interpretation that the antisymmetric part of $\mathbf{W}$ is the bivector spin velocity of rigid director frames.

It also follows as a consequence of equations (2.8) and (2.10) that for rigid directors $W_{(ij);k} = 0$, and that $W_{[ij];k}$ is the gradient of the director spin bivector. We recall from equation (2.6) that the symmetric part of $\mathbf{W}$ is a measure of relative changes in the lengths of, and the angles between, the directors.

Support for these results is found in several previous works. Grad (1952), from energy considerations, obtains the result that, for rigid particles, couple stress corresponds to bivector spin gradients. Dahler (1965) obtains this result using a statistical mechanical approach. Mindlin (1964) gives an excellent physical description of the twenty-seven components of orientation strain in a solid with microstructure from a slightly different point of view.

Since we have restricted our analysis to a rigid microstructure, we are only concerned with the twelve components represented by $W_{[mn]}$ and $W_{[mn];k}$.

2.3. *Balance equations*

We record the balance equations for mass, linear momentum and total angular momentum when the microstructure is non-deforming (see, for example, Toupin 1962, 1964; Dahler & Scriven 1963; Condiff & Dahler 1964; Mindlin 1964; Eringen 1964),

$$\left.\begin{aligned} \dot{\rho} + \rho v^k_{;k} &= 0, \\ t^{ki}_{;k} + \rho f^i &= \rho \dot{v}^i, \\ \mu^{ki}_{;k} + \rho l^i - \epsilon^{ijk} t_{kj} &= \rho\dot{\sigma}^i, \end{aligned}\right\} \tag{2.11}$$

where ρ is the mass density, t^{ij} and μ^{ij} are the contravariant components of the Cauchy stress $\mathbf{t}$ and the couple stress $\boldsymbol{\mu}$ respectively, $\mathbf{l}$ is the body couple, and $\boldsymbol{\sigma}$ is the spin momentum density.

3. Constitutive equations

3.1. *General constitutive relations for a Cosserat fluid*

The rate of strain of position is measured by the velocity gradients $v_{i;j}$, whereas the rate of strain of orientation for a rigid microstructure is measured by $W_{[ij]}$ and $W_{[ij];k}$ or equivalently by the spin velocity bivector $\Omega_{[ij]}$ and its gradient $\Omega_{[ij];k}$. However, since we describe the couple stress by the second-order tensor μ^{ij}, we may replace $\Omega_{[ij]}$ by the equivalent vector Ω_i.

It is convenient at this point to work in Cartesian tensor notation. We take as constitutive equations for a Cosserat fluid (i.e. a fluid with rigid directors)

$$\left.\begin{aligned} t_{ij} &= f_{ij}(v_{k,m};\, \Omega_{k,};\, \Omega_{k,m};\, \mathbf{d}_\alpha), \\ \mu_{ij} &= h_{ij}(v_{k,m};\, \Omega_{k,};\, \Omega_{k,m};\, \mathbf{d}_\alpha), \end{aligned}\right\} \tag{3.1}$$

where we have used the principle of equipresence and where the comma before a subscript denotes partial differentiation.

We examine $v_{k,m}$, Ω_k, $\Omega_{k,m}$ to find objective forms of these arguments under the rigid transformation

$$\hat{x}_i = Q_{ij}(t)\,x_j + b_i(t), \tag{3.2}$$

where

$$Q_{ij}Q_{jk} = \delta_{ik}. \tag{3.3}$$

Taking the time derivative of equation (3.2) yields

$$\hat{v}_k = Q_{kj}v_j + \lambda_{kr}Q_{rj}x_j + \dot{b}_k,$$

where

$$\dot{Q}_{ij} = \lambda_{ik}Q_{kj}, \quad \lambda_{ik} = \lambda_{[ik]} = \dot{Q}_{ij}Q_{jk}.$$

Consequently with the use of equation (3.3)

$$\hat{v}_{k,m} = \partial\hat{v}_k/\partial\hat{x}_m = Q_{kj}Q_{nm}v_{j,n} + \lambda_{km},$$

$$\hat{d}_{km} = Q_{kj}Q_{mn}d_{jn} \quad \text{and} \quad \hat{\omega}_{km} = Q_{kj}Q_{mn}\omega_{jn} + \lambda_{km},$$

where $d_{km} = v_{(k,m)}$ and $\omega_{km} = v_{[k,m]}$. We now choose $Q_{ij} = \delta_{ij}$ and, since each point of the fluid has associated with it the rigid rotation $\mathbf{\Omega}$, we set $\lambda_{km} = \epsilon_{mkr}\Omega_r$. The last set of equations then yield

$$\hat{d}_{km} = d_{km}, \tag{3.4}$$

$$\hat{\omega}_{km} = \omega_{km} + \epsilon_{mkr}\Omega_r. \tag{3.5}$$

Similarly

$$\hat{\Omega}_k = 0 \quad \text{and} \quad \hat{\Omega}_{k,m} = \frac{\partial\hat{\Omega}_k}{\partial\hat{x}_m} = \frac{\partial\Omega_k}{\partial x_m}Q_{nm} = \Omega_{k,m}. \tag{3.6}$$

Then

$$\left.\begin{aligned} t_{ij} &= f_{ij}(d_{km} + \omega_{km} + \epsilon_{mkr}\Omega_r;\, \Omega_{k,m};\, \mathbf{d}_\alpha), \\ \mu_{ij} &= h_{ij}(d_{km} + \omega_{km} + \epsilon_{mkr}\Omega_r;\, \Omega_{k,m};\, \mathbf{d}_\alpha). \end{aligned}\right\} \tag{3.7}$$

Results (3.4) and (3.5) were deduced earlier by Born (1920) from physical considerations. The results expressed by equations (3.7) were obtained by Dahler (1965) from statistical mechanics. Grad (1952) deduced the general form of the constitutive equations by considering entropy production.

3.2. *Explicit constitutive equations for transverse isotropy*

We restrict ourselves at this point to fluids with substructures exhibiting a single director vector which will be denoted by $\mathbf{n}$; i.e. the index α takes only the value 1, $\mathbf{d}_1 = \mathbf{n}$. Thus, for example, $\mathbf{n}$ could represent the orientation of the axis of revolution for a body of revolution. Furthermore, since we have assumed that the substructure does not deform under the influence of the surrounding fluid, we require $\mathbf{n}$ to be of constant length, i.e.

$$n_i n_i = n^2. \tag{3.8}$$

We assume an incompressible fluid such that

$$v_{k,k} = d_{kk} = 0. \tag{3.9}$$

We postulate linear cause–effect relations, taking

$$\left.\begin{aligned} t_{ij} &= A_{ij} + B_{ijkm}(d_{km} + \omega_{km} + \epsilon_{mkr}\Omega_r) + C_{ijkm}\Omega_{k,m}, \\ \mu_{ij} &= D_{ijkm}\Omega_{k,m} + E_{ijkm}(d_{km} + \omega_{km} + \epsilon_{mkr}\Omega_r), \end{aligned}\right\} \tag{3.10}$$

where **A**, **B**, **C**, **D** are phenomenological coefficients which are functions of the density ρ, the temperature T and the single preferred direction $\mathbf{n}$.

Following Ericksen (1960*a*, *b*), we assume that these relations are invariant under reflexions through all planes containing $\mathbf{n}$ and that $\mathbf{n}$ and $-\mathbf{n}$ are physically indistinguishable. These restrictions require that **A**, **B**, **C** and **D** be transversely isotropic tensors with respect to the direction $\mathbf{n}$. Any such tensor is expressible as a linear combination of outer products of n_i and δ_{ij} and the scalar coefficients reduce to functions of ρ, T, and n^2. In the sequel we will consider $\mathbf{n}$ to be normalized since this entails no loss of generality. We emphasize, however, that the scalar coefficients are dependent on density, temperature and some scalar representative of particle size.

It follows that

$$A_{ij} = \gamma_0 \delta_{ij} + \gamma_1 n_i n_j, \tag{3.11}$$

and

$$\begin{aligned} B_{ijkm} = {} & \gamma_2 \delta_{ij} \delta_{km} + \gamma_3 \delta_{ik} \delta_{jm} + \gamma_4 \delta_{im} \delta_{jk} \\ & + \gamma_5 n_i n_j \delta_{km} + \gamma_6 \delta_{ij} n_k n_m + \gamma_7 \delta_{ik} n_j n_m \\ & + \gamma_8 n_i n_k \delta_{jm} + \gamma_9 n_i n_m \delta_{jk} + \gamma_{10} \delta_{im} n_j n_k \\ & + \gamma_{11} n_i n_j n_k n_m. \end{aligned} \tag{3.12}$$

Similar expressions can be written for **C**, **D** and **E** with different scalar coefficients replacing γ_k in equation (3.12).

As postulated by Grad (1952) and Dahler (1965) we assume that the presence of $\Omega_{k,m}$ in the traction stress equation and of $(v_{k;m} + \epsilon_{mkr}\Omega_r)$ in the couple stress equation represents higher-order effects. Ignoring higher-order effects and assuming $\boldsymbol{\mu} = 0$ when $\nabla\boldsymbol{\Omega} = 0$, we may write the explicit constitutive equations as:

$$\left.\begin{aligned} t_{ij} = {} & -p\delta_{ij} + \alpha_1 n_i n_j + 2\alpha_2 d_{ij} + 2\alpha_3 d_{jk} n_k n_i + 2\alpha_4 d_{ik} n_k n_j \\ & + \alpha_5 d_{km} n_k n_m n_i n_j + 2\alpha_6(\omega_{ij} + \epsilon_{jik}\Omega_k) + 2\alpha_7(\omega_{ik} + \epsilon_{kir}\Omega_r) n_k n_j \\ & + 2\alpha_8(\omega_{jk} + \epsilon_{kjr}\Omega_r) n_k n_i, \\ \mu_{ij} = {} & \beta_1 \Omega_{k,k} \delta_{ij} + \beta_2 \Omega_{i,j} + \beta_3 \Omega_{j,i} + \beta_4 \Omega_{k,k} n_i n_j \\ & + \beta_5 \Omega_{k,m} n_k n_m \delta_{ij} + \beta_6 \Omega_{i,k} n_k n_j + \beta_7 \Omega_{k,j} n_k n_i \\ & + \beta_8 \Omega_{j,k} n_k n_i + \beta_9 \Omega_{k,i} n_k n_j + \beta_{10} \Omega_{k,m} n_k n_m n_i n_j. \end{aligned}\right\} \tag{3.13a,b}$$

If the microstructure exhibits no preferred direction, i.e. if it is spherical, then on setting $n_i = 0$ we recover the constitutive equations for an isotropic fluid with microstructure. Note that

$$2(\omega_{ij} + \epsilon_{jik}\Omega_k) = \mathbf{U} \wedge (\operatorname{curl} \mathbf{v} - 2\boldsymbol{\Omega}), \tag{3.14}$$

where **U** is the identity tensor. Equation (3.14) leads to the physical interpretation that the antisymmetric part of the stress tensor arises as a consequence of the difference between fluid vorticity and spin.

Finally, we emphasize that the conservation equations (2.11) were written for the special case of a non-deforming or rigid microstructure. This permits us to write the expression for the spin momentum as

$$\boldsymbol{\sigma} = \mathbf{I} . \boldsymbol{\Omega}, \tag{3.15}$$

where $\mathbf{I}$ is the mass momentum of inertia per unit mass, assumed to be a symmetric second-order tensor. Moreover, for rigid particles

$$\dot{\mathbf{I}} = \mathbf{\Omega}\wedge\mathbf{I}-\mathbf{I}\wedge\mathbf{\Omega} \tag{3.16}$$

gives the rate of change of the inertia tensor associated with co-ordinates fixed in a rotating fluid particle.

A count gives as unknowns ρ, v_i, Ω_i, n_i, t_{ij}, μ_{ij}, I_{ij} or a total of thirty-four, while we have conservation of mass and momenta, traction and couple stress relations and equation (3.16) (relating moment of inertia to angular velocity) yielding a total of thirty-one equations. In order to have a determinate problem, three additional equations are required. These are provided by the expression

$$\dot{\mathbf{n}} = \mathbf{\Omega}\times\mathbf{n}, \tag{3.17}$$

relating angular velocity to the orientation vector $\mathbf{n}$.

We point out that for bodies of revolution, if $\mathbf{n}$ describes the orientation of the axis of revolution, equation (3.16) can be replaced by

$$I_{ij} = (I_1-I_2)\,n_i n_j + I_2\delta_{ij}, \tag{3.18}$$

where I_1 and I_2 are the principal moments of inertia.

The reduction to a theory of negligible microstructure interaction (which may be interpreted, for example, as a dilute suspension) is accomplished by setting the flux of angular momentum equal to zero. We are therefore defining a Cosserat fluid with negligible substructure interaction as satisfying the requirement $\nabla\mathbf{\Omega} = \boldsymbol{\mu} = 0$. The requirement for non-negligible substructure interaction is $\nabla\mathbf{\Omega} \neq 0$, $\boldsymbol{\mu} \neq 0$.

4. Equations for rectilinear shearing motion

In this section we exhibit the complete system of equations governing the special case of rectilinear shearing motion. We choose a rectangular Cartesian co-ordinate system in which x_1 is parallel to the direction of motion, x_2 is parallel to the direction of the velocity gradient and the system is right-handed.

A steady shearing motion is assumed in which there is no dependence on x_1 and x_3,

$$\left.\begin{aligned} v_i &= (v_1(x_2), 0, 0),\\ n_i &= (n_1(x_2,t), n_2(x_2,t), n_3(x_2,t)),\end{aligned}\right\} \tag{4.1}$$

and there is negligible substructure spin inertia, i.e. $\rho\ddot{n}_i = 0$.

Using equations (3.13 a) and (4.1), we record the following typical components of $\mathbf{t}$ and $\boldsymbol{\mu}$:

$$\left.\begin{aligned} t_{11} = {} & -p+\alpha_1 n_1^2+(\alpha_3+\alpha_4+\alpha_7+\alpha_8)\,n_1 n_2 v_{1,2}+\alpha_5 n_1^3 n_2 v_{1,2}\\ & +2(\alpha_7+\alpha_8)\,n_1 n_2\,\Omega_3-2(\alpha_7+\alpha_8)\,n_1 n_3\,\Omega_2,\\ t_{12} = {} & a_1 n_1 n_2+(\alpha_2+\alpha_6)\,v_{1,2}+(\alpha_3-\alpha_8)\,n_1^2 v_{1,2}+(\alpha_4+\alpha_7)\,n_2^2 v_{1,2}\\ & +\alpha_5 n_1^2 n_2^2 v_{1,2}+2\alpha_6\,\Omega_3+2\alpha_7 n_2^2\,\Omega_3-2\alpha_8 n_1^2\,\Omega_3-2\alpha_7 n_2 n_3\,\Omega_2\\ & +2\alpha_8 n_1 n_3\,\Omega_1,\end{aligned}\right\} \tag{4.2}$$

$$\left.\begin{aligned}\mu_{11} &= [n_1 n_2(\beta_5+\beta_6+\beta_8+n_1^2\beta_{10})]\,\Omega_{1,2}+[\beta_1+\beta_4 n_1^2+\beta_5 n_2^2+\beta_{10}n_1^2 n_2^2]\,\Omega_{2,2}\\ &\quad+[\beta_5 n_2 n_3+\beta_{10}n_1^2 n_2 n_3]\,\Omega_{3,2},\\ \mu_{12} &= [\beta_2+\beta_6 n_2^2+\beta_7 n_1^2+\beta_{10}n_1^2 n_2^2]\,\Omega_{1,2}+[n_1 n_2(\beta_4+\beta_7+\beta_8+\beta_{10}n_2^2)]\,\Omega_{2,2}\\ &\quad+[n_1 n_3(\beta_7+\beta_{10}n_2^2)]\,\Omega_{3,2}.\end{aligned}\right\}\tag{4.3}$$

The remaining components of t_{ij} and μ_{ij} may be similarly derived. Noting that the continuity equation is identically satisfied, we use equations (4.1), (4.2) and (4.3) to write the x_1-components of the momentum equations (2.10) in the form

$$\begin{aligned}-\frac{\partial}{\partial x_1}p+\frac{\partial}{\partial x_2}&[\alpha_1 n_1 n_2+(\alpha_2-\alpha_6)\,v_{1,2}+(\alpha_3+\alpha_8)\,n_2^2 v_{1,2}\\ &+(\alpha_4-\alpha_7)\,n_1^2 v_{1,2}+\alpha_5 n_1^2 n_2^2 v_{1,2}-2\alpha_6\,\Omega_3-2\alpha_7 n_1^2\,\Omega_3\\ &+2\alpha_8 n_2^2\,\Omega_3+2\alpha_7 n_1 n_3\,\Omega_1-2\alpha_8 n_2 n_3\,\Omega_2] = 0,\end{aligned}\tag{4.4}$$

$$\begin{aligned}\frac{\partial}{\partial x_2}&[(\beta_3+\beta_8 n_2^2+\beta_9 n_1^2+\beta_{10}n_1^2 n_2^2)\,\Omega_{1,2}\\ &+n_1 n_2(\beta_4+\beta_6+\beta_9+\beta_{10}n_2^2)\,\Omega_{2,2}+n_1 n_3(\beta_9+\beta_{10}n_2^2)\,\Omega_{3,2}]\\ &+(\alpha_4-\alpha_3-\alpha_7+\alpha_8)\,n_1 n_3 v_{1,2}+4\alpha_6\,\Omega_1+2(\alpha_7-\alpha_8)\,(n_2^2+n_3^2)\,\Omega_1\\ &-2(\alpha_7-\alpha_8)\,n_1 n_3\,\Omega_3-2(\alpha_7-\alpha_8)\,n_1 n_2\,\Omega_3 = \rho\dot{\sigma}_1.\end{aligned}\tag{4.5}$$

The remaining components of equations (2.10) have a similar structure. Note that equations (4.4) and (4.5) are consistent with assumption (4.1) only if $n_3 = 0$ or if $n_{i,j} = \Omega_{i,j} = 0$. Otherwise rectilinear shearing motion of the type specified by equation (4.1) cannot exist.

5. Rectilinear shearing motion with $\nabla\Omega = 0$

When the mutual interactions of the fluid microstructure are negligible, the flux of spin velocity is small and we set $\Omega_{i,j} = \mu_{ij} = 0$. We take this to represent the flow of a dilute suspension, since, if the concentration of suspended particles is low, the effect of particle interaction will be weak. As previously mentioned, dilute fluid suspensions have been examined by Jeffery (1922) and Ericksen (1960*b*). In order to compare our results with theirs, we make the following additional assumptions:

$$v_i = (kx_2, 0, 0), \quad p_{,i} = n_{i,j} = 0. \tag{5.1}$$

Thus, no mechanical pressure gradients are imposed, and the n_i are completely independent of position.

From (5.1) and (4.4) we see that the equations of linear momentum are identically satisfied. However, we have yet to satisfy the spin momentum equations (4.5). If we neglect spin of the particle about its axis of symmetry, i.e. we set

$$\Omega_n = \mathbf{\Omega}.\mathbf{n} = 0, \tag{5.2}$$

we can then compare our results directly with Ericksen's (1960*b*) since his theory does not account for the spin component Ω_n. Now, from equations (3.17) and (5.2),

$$\mathbf{n}\wedge\dot{\mathbf{n}} = \mathbf{\Omega}. \tag{5.3}$$

Using equations (5.1) and (5.3), equation (4.5) reduces to the form

$$\left.\begin{aligned} n_2\dot{n}_3 - n_3\dot{n}_2 &= an_1n_3, \\ n_3\dot{n}_1 - n_1\dot{n}_3 &= bn_2n_3, \\ n_1\dot{n}_2 - n_2\dot{n}_1 &= c-(an_1^2+bn_2^2), \end{aligned}\right\} \tag{5.4}$$

where, for conciseness in notation, we have set

$$a = \frac{k(\alpha_3-\alpha_4+\alpha_7-\alpha_8)}{4\alpha_6+2(\alpha_7-\alpha_8)}, \quad b = \frac{k(-\alpha_3+\alpha_4+\alpha_7-\alpha_8)}{4\alpha_6+2(\alpha_7-\alpha_8)}, \quad c = \frac{-2k\alpha_6}{4\alpha_6+2(\alpha_7-\alpha_8)}. \tag{5.5}$$

We can describe the orientation of $\mathbf{n}$ in space by means of two angles. Let θ represent the angle from the x_1-axis to the projection of $\mathbf{n}$ on the x_1–x_3 plane. Let ϕ be the angle from this same projection to $\mathbf{n}$. Then

$$n_1 = \cos\theta\cos\phi, \quad n_2 = \sin\phi, \quad n_3 = \sin\theta\cos\phi. \tag{5.6}$$

Substituting (5.6) into (5.4) gives

$$\left.\begin{aligned} \sin\phi\cos\phi\cos\theta\dot{\theta} - \sin\theta\dot{\phi} &= a\sin\theta\cos\theta\cos^2\phi, \\ \cos\phi\dot{\theta} &= -b\sin\theta\sin\phi, \\ \sin\theta\sin\phi\cos\phi\dot{\theta} + \cos\theta\dot{\phi} &= c-(a\cos^2\theta\cos^2\phi + b\sin^2\phi). \end{aligned}\right\} \tag{5.7}$$

Note that one of equations (5.7) is redundant since $\mathbf{n}$ satisfies equation (3.8). There is more than one possible solution of system (5.7).

Case 1

One possible solution is given by

$$\sin\theta = 0, \quad \cos\theta = \pm 1. \tag{5.8}$$

Then system (5.7) becomes

$$\theta = \dot{\theta} = 0, \quad \pm\dot{\phi} = c-(a\cos^2\phi + b\sin^2\phi), \tag{5.9}$$

i.e. $n_3 = 0$, and the substructure particles rotate in the x_1–x_2 plane with the motion given by (5.9). This result can be put into a simpler form in terms of the constant

$$\gamma = \frac{\alpha_3-\alpha_4}{2\alpha_6+\alpha_7-\alpha_8}. \tag{5.10}$$

Then,

$$\dot{\phi} = \pm\tfrac{1}{2}k[1+\gamma(\cos^2\phi - \sin^2\phi)],$$

which has as solutions

$$\left.\begin{aligned} \phi &= \tan^{-1}\left[\left(\frac{1+\gamma}{1-\gamma}\right)^{\frac{1}{2}}\tan\{\tfrac{1}{2}k(1-\gamma^2)^{\frac{1}{2}}t\}\right] \quad (\gamma^2<1), \\ \frac{\gamma+\cos 2\phi + (\gamma^2-1)^{\frac{1}{2}}\sin 2\phi}{1+\gamma\cos 2\phi} &= \exp[-k(\gamma^2-1)^{\frac{1}{2}}t] \quad (\gamma^2>1). \end{aligned}\right\} \tag{5.11a,b}$$

We now consider the two possibilities, $\gamma^2 \geqslant 1$, $\gamma^2 < 1$.

(*a*) $\gamma^2 \geqslant 1$

From equation (5.11*b*) we see that as the solution approaches steady state, $t \to \infty$, the orientation of the substructure particles tends towards the constant value given by

$$\gamma + \cos 2\phi + (\gamma^2-1)^{\frac{1}{2}}\sin 2\phi = 0. \tag{5.12}$$

As γ becomes large, ϕ approaches 45°; i.e. the vector $\mathbf{n}$ is aligned at an angle of 45° to the streamlines. As $\gamma \to -1$, the angle ϕ approaches zero; i.e. the vector $\mathbf{n}$ becomes aligned parallel to the fluid streamlines.

Ericksen (1960*b*) obtained the constant solution (5.12) with $n_3 = 0$ and the particles oriented between 0° and 45° with the fluid streamlines. He also obtained an unstable solution, $n_1 = n_2 = 0$, $n_3 = \pm 1$, which is not possible with our formulation.

The steady-state stresses for this case are given by

$$\left.\begin{aligned} t_{11} &= -p + \alpha_1 n_1^2 + (\alpha_3 + \alpha_4 + \alpha_7 + \alpha_8) n_1 n_2 k + \alpha_5 n_1^3 n_2 k, \\ t_{22} &= -p + \alpha_1 n_2^2 + (\alpha_3 + \alpha_4 - \alpha_7 - \alpha_8) n_1 n_2 k + \alpha_5 n_1 n_2^3 k, \\ t_{33} &= -p, \\ t_{12} &= \alpha_1 n_1 n_2 + (\alpha_2 + \alpha_6) k + (\alpha_3 - \alpha_8) n_1^2 k \\ &\qquad + (\alpha_4 + \alpha_7) n_2^2 k + \alpha_5 n_1^2 n_2^2 k, \\ t_{21} &= \alpha_1 n_1 n_2 + (\alpha_2 - \alpha_6) k + (\alpha_3 + \alpha_8) n_2^2 k \\ &\qquad + (\alpha_4 - \alpha_7) n_1^2 k + \alpha_5 n_1^2 n_2^2 k. \end{aligned}\right\} \tag{5.13}$$

These results are similar in form to those found by Ericksen (1960*b*) except that the shear stress components in (5.13) are not equal. Thus, this theory exhibits not only the non-Newtonian effects predicted by Ericksen (viz. the Weissenberg effect, a decrease in apparent viscosity t_{12}/k with increasing shear rate), but also the additional effect of a non-symmetric stress tensor.

(*b*) $\gamma^2 < 1$

Consideration of equation (5.11*a*) reveals that $\mathbf{n}$ varies periodically with time and is of the form obtained originally by Jeffery (1922). The angular acceleration at any instant is not zero, but the net angular acceleration over a period is zero. However, when $\gamma = 0$, i.e. $\alpha_3 = \alpha_4$,

$$\phi = \tfrac{1}{2}kt, \quad \gamma = 0, \tag{5.14}$$

and the substructure particles rotate with constant angular velocity equal to one-half the shear rate. Thus, the spin velocity is nothing more than the fluid vorticity. This leads us to the same observation made by Ericksen (1960*b*), that γ is a measure of the substructure particle eccentricity. When the eccentricity becomes zero, the substructure becomes isotropic, i.e. spherical, and the spin velocity at every point of the fluid becomes equal to the fluid vorticity. This constant angular velocity was obtained by Jeffery (1922) for a spheroidal particle and, hence, the case $n_3 = 0, \gamma = 0$ in this work corresponds to his result. However, for $0 < \gamma^2 < 1$, we see from equation $(5.11)_1$ that the present theory allows the substructure to have an angular acceleration at any instant with the net acceleration over the period of rotation equal to zero.

Clearly the stresses for this case vary periodically with time. As suggested by Ericksen (1960*b*), one can obtain 'expected values' of stress by averaging over a period. Integrating equations (4.2) over the period yields for $\gamma = 0$

$$\left.\begin{aligned} \bar{t}_{11} = -\bar{p} + \tfrac{1}{2}\alpha_1, \quad \bar{t}_{22} = -\bar{p} + \tfrac{1}{2}\alpha_1, \quad \bar{t}_{33} = -\bar{p}, \\ \bar{t}_{12} = \alpha_1^* k, \quad \bar{t}_{21} = \alpha_2^* k, \end{aligned}\right\} \tag{5.15}$$

where α_1^* and α_2^* are simply groups of physical constants. It is evident from (4.2) that if $\alpha_1 = 0$, i.e. the fluid is isotropic at rest, the stresses (5.15) are indistinguishable from the Navier–Stokes stresses except for the non-symmetry of t_{ij} (see Ericksen 1960*b*).

Case 2

Another solution of the system (5.7) is possible for $\sin\theta \neq 0$. Then

$$\dot{\phi} = -\cos\theta\,(a\cos^2\phi + b\sin^2\phi), \quad \dot{\theta} = -b\sin\theta\tan\phi.$$

In the phase plane

$$\frac{d\theta}{b\tan\theta} = \frac{\tan\phi\, d\phi}{a\cos^2\phi + b\sin^2\phi}. \tag{5.16}$$

If we make the substitution $y = \tan\phi$ on the right side of (5.16) and integrate, we obtain

$$C^*|\sin\theta| = |b\tan^2\phi + a|^{\frac{1}{2}}, \tag{5.17}$$

where C^* is a constant of integration.

Examination of equation (5.17) reveals that the vector $\mathbf{n}$ describes a cone about the x_1-axis. The cone half angle measured in the x_1–x_2 plane is given by (5.17) evaluated at $\theta = 0$,

$$\phi|_{\theta=0} = \pm\tan^{-1}(a/b)^{\frac{1}{2}}. \tag{5.18}$$

The cone half angle measured in the x_1–x_3 plane is given by (5.17) evaluated at $\phi = 0$,

$$\theta|_{\phi=0} = \pm\sin^{-1}(|a|/C^{*2})^{\frac{1}{2}}. \tag{5.19}$$

This solution exhibits a type of motion found possible by Jeffery (1922) and by Ericksen (1960*b*). We conclude that the present theory when specialized to the case of negligible particle interaction yields results compatible with previous work on dilute suspensions.

Finally, we note that, when the assumptions are made simultaneously that $\Omega_n = 0$ and $n_3 \neq 0$, it is necessary, in order for equations (5.4) or (5.7) to be self-consistent, to set $c = 0$ and hence $\alpha_6 = 0$.

6. Rectilinear shearing motion with $\nabla\Omega \neq 0$

6.1. *Introductory remarks*

In this section we will consider the problem of rectilinear shearing motion for a fluid in which the mutual interactions of the substructure are no longer negligible. The case of fluid suspensions will be used as an example to discuss some of the non-Newtonian effects observed experimentally. In the case of blood flow these non-Newtonian effects are attributed to the presence of red cells in suspension. The red cells are biconcave disks and are bodies of revolution whose largest dimension is about 8·5μm. The blood vessel diameters vary from 2·5 cm to 8μm. The theory presented would apply only where the fluid passage is much larger than the largest dimension of the suspended particles. When particle size is of the same order of magnitude as the fluid passage, the deformation of suspended particles will become important and the present continuum approach may be invalid.

The more obvious non-Newtonian effects observed experimentally by blood rheologists are (Haynes 1961):

(1) The 'apparent viscosity' defined by $t_{12}/v_{1,2}$ decreases with increasing shear rate $v_{1,2}$.

(2) The size of suspended particles as compared with a characteristic length (fluid passage size) of a specific problem affects the apparent viscosity. For a given particle size, decreasing passage size decreases apparent viscosity.

(3) Apparent viscosity increases with increasing volume concentration of suspended particles.

(4) In Poiseuille flow, an axial accumulation of suspended particles is observed and a marginal zone near the walls is observed which is deficient in suspended particles. This is attributed by Haynes (1961) to a transverse 'Magnus force arising from particle rotation in a variable shear field.' This so-called Magnus force has not been measured but has been proposed to explain the particle concentration gradients.

Kinematical results for suspensions might also be expected to depend on the deviation of the suspended particles from the spherical shape. Indeed, the theoretical results obtained in the previous section and by Ericksen (1960*b*) for dilute suspensions indicate that this is the case. We would also expect this effect to be present in the non-dilute case. As noted in § 5, the dilute theory accounts for variation of apparent viscosity with shear rate and for variation in particle shape. However, a more refined theory is required to explain other non-Newtonian phenomena especially for particle concentrations such as those encountered in blood flow (40 % by volume).

In order to exhibit some of the effects of concentration on the flow kinematics without resorting to numerical techniques, we restrict ourselves to the case

$$n_i = (n_1/(x_2,t), n_2(x_2,t), 0), \quad \Omega_i = (0,0,\Omega_3(x_2)). \tag{6.1}$$

This example was previously considered by Condiff & Dahler (1964) for the special case of an isotropic microstructure, i.e. $n_i = 0$, using the equations developed by Dahler (1965) from statistical mechanical considerations.

6.2. *Governing equations*

The equations of motion (4.4) and (4.5), under the assumptions of equation (6.1), take the form

$$\left.\begin{aligned}
&-\frac{\partial}{\partial x_1}p+\frac{\partial}{\partial x_2}[\alpha_1 n_1 n_2+(\alpha_2-\alpha_6)v_{1,2}+(\alpha_3+\alpha_8)n_2^2 v_{1,2}\\
&\qquad +(\alpha_4-\alpha_7)n_1^2 v_{1,2}+\alpha_5 n_1^2 n_2^2 v_{1,2}-2\alpha_6\Omega_3-2\alpha_7 n_1^2\Omega_3+2\alpha_8 n_2^2\Omega_3]=0,\\
&\frac{\partial}{\partial x_2}[-p+\alpha_1 n_2^2+(\alpha_3+\alpha_4-\alpha_7-\alpha_8)n_1 n_2 v_{1,2}\\
&\qquad\qquad +\alpha_5 n_1 n_2^3 v_{1,2}-2(\alpha_7+\alpha_8)n_1 n_2\Omega_3]=0,\\
&\qquad\qquad\qquad -\frac{\partial}{\partial x_3}p=0,\\
&\frac{\partial}{\partial x_2}[(\beta_3+\beta_8 n_2^2)\Omega_{3,2}]+(2\alpha_6+\alpha_7-\alpha_8)v_{1,2}\\
&\qquad +(\alpha_3-\alpha_4)(n_1^2-n_2^2)v_{1,2}+2(2\alpha_6+\alpha_7-\alpha_8)\Omega_3=0.
\end{aligned}\right\} \tag{6.2a,b,c,d}$$

Two types of solution are possible for system (6.2). They are:

(*a*) Solution of the first type:

$$\left.\begin{aligned} v_{1,2} = k. \quad \Omega_3 = \dot{\phi} = 0, \\ \frac{\partial}{\partial x_i} p = (0,0,0), \quad \frac{\partial n_i}{\partial x_j} = 0, \\ n_1 = \cos\phi, \quad n_2 = \sin\phi, \quad n_3 = 0, \\ 2\phi = \cos^{-1} 1/\gamma, \quad \gamma^2 \geqslant 1. \end{aligned}\right\} \tag{6.3}$$

In this case, there is no motion of the substructure particles and the preferred direction $\mathbf{n}$ is oriented between 0° and 45° with the fluid streamlines depending on the value of the physical constant γ defined in (5.10). This is the same solution as for the dilute suspension case 1 (*a*) of § 5.

(*b*) Solution of the second type:

Assume that

$$\Omega_3 = \dot{\phi} \neq 0. \tag{6.4}$$

We must then solve the full set of equations (6.2) which is the non-dilute counterpart of case 1 (*b*) of § 5. The particles are rotating with constant angular velocity $\Omega_3(x_2)$. Equations (6.2) become more tractable when averaged over a particle period T. Hence, we define

$$\bar{F} = \frac{1}{T}\int_0^T F(x_2, t)\, dt. \tag{6.5}$$

Stipulating that $p_{,1} = 0$, we integrate the first three equations of system (6.2) with respect to x_2 and obtain

$$\left.\begin{aligned} p = \alpha_1 n_2^2 + (\alpha_3+\alpha_4-\alpha_7-\alpha_8)\, n_1 n_2 v_{1,2} \\ + \alpha_5 n_1 n_2^3 v_{1,2} - 2(\alpha_7+\alpha_8) n_1 n_2 \Omega_3 + A, \\ \alpha_1 n_1 n_2 + (\alpha_2 - \alpha_6)\, v_{1,2} + (\alpha_3+\alpha_8)\, n_2^2 v_{1,2} \\ + (\alpha_4-\alpha_7)\, n_1^2 v_{1,2} + \alpha_5 n_1^2 n_2^2 v_{1,2} - 2\alpha_6 \Omega_3 \\ - 2\alpha_7 n_1^2 \Omega_3 + 2\alpha_8 n_2^2 \Omega_3 = B, \\ \frac{\partial}{\partial x_2}\,[(\beta_3+\beta_8 n_2^2)\,\Omega_{3,2}] + (2\alpha_6+\alpha_7-\alpha_8)\, v_{1,2} \\ + (\alpha_3-\alpha_4)\,(n_1^2-n_2^2)\, v_{1,2} + 2(2\alpha_6+\alpha_7-\alpha_8)\,\Omega_3 = 0, \end{aligned}\right\} \tag{6.6a,b,c}$$

where A and B are constants of integration. Note that for a Newtonian fluid

$$p = p_0 = \text{const.} \tag{6.7}$$

From (6.6, a)

$$\begin{aligned} p = p_0 + \alpha_1 n_2^2 + (\alpha_3+\alpha_4-\alpha_7-\alpha_8)\, n_1 n_2 v_{1,2} \\ + \alpha_5 n_1 n_2^3 v_{1,2} - 2(\alpha_7+\alpha_8)\, n_1 n_2 \Omega_3. \end{aligned} \tag{6.8}$$

We can interpret $(p-p_0)$ in (6.8) as a 'transverse Magnus force' which we denote by p_M. Now

$$\frac{\partial p_M}{\partial x_2} = \frac{\partial}{\partial x_2}\,[\alpha_1 n_2^2 + (\alpha_3+\alpha_4-\alpha_7-\alpha_8)\, n_1 n_2 v_{1,2} + \alpha_5 n_1 n_2^3 v_{1,2} - 2(\alpha_7+\alpha_8)\, n_1 n_2 \Omega_3]. \tag{6.9}$$

We apply the definition (6.5) to equations $(6.6)_2$, $(6.6)_3$ and (6.9) to obtain

$$\left.\begin{aligned}(2\alpha_2-2\alpha_6+\alpha_3+\alpha_4-\alpha_7+\alpha_8+\tfrac{1}{4}\alpha_5)\,v_{1,2}-2(2\alpha_6+\alpha_7-\alpha_8)\,\Omega_3 &= 2B,\\ (2\beta_3+\beta_8)\,\Omega_{3,22}-\beta_8\frac{\Omega_{3,2}^2}{\Omega_3}+4(2\alpha_6+\alpha_7-\alpha_8)\,\Omega_3+2(2\alpha_6+\alpha_7-\alpha_8)\,v_{1,2} &= 0,\\ \frac{\overline{\partial p_M}}{\partial x_2} = -\frac{\alpha_1}{2}\frac{\Omega_{3,2}}{\Omega_3}+\frac{\epsilon_1}{2}\frac{\Omega_{3,2}}{\Omega_3}v_{1,2}(\pi+\tfrac{1}{2})-\epsilon_2\,\Omega_{3,2}(\pi+\tfrac{1}{2}),&\end{aligned}\right\}\quad (6.10a,b,c)$$

where
$$\epsilon_1 = \alpha_3+\alpha_4-\alpha_7-\alpha_8, \quad \epsilon_2 = \alpha_7+\alpha_8. \qquad (6.11)$$

Equations $(6.10a)$ and $(6.10b)$ govern the motion of the fluid while $(6.10c)$ gives the expression for the gradient in the x_2 direction of the Magnus force averaged over a particle period. In a similar manner we can obtain the 'expected' traction and couple stresses corresponding to (5.15). For example, the average couple stresses from (4.3) are:

$$\left.\begin{aligned}&\overline{\mu}_{11} = \overline{\mu}_{22} = \overline{\mu}_{33} = \overline{\mu}_{12} = \overline{\mu}_{21} = \overline{\mu}_{13} = \overline{\mu}_{31} = 0,\\ &\overline{\mu}_{23} = (\beta_3+\tfrac{1}{2}\beta_8)\,\Omega_{3,2},\\ &\overline{\mu}_{32} = (\beta_2+\tfrac{1}{2}\beta_6)\,\Omega_{3,2}.\end{aligned}\right\}\qquad (6.12)$$

Henceforth we restrict attention to the equations of motion $(6.10a)$ and $(6.10b)$. We now consider the rectilinear shearing motion as representing the flow between plates parallel to the x_1–x_3 plane at a distance h apart with the lower plate fixed at $x_2 = 0$. The upper plate at $x_2 = h$ moves with constant velocity V. Setting

$$y = x_2/h, \quad u = v_1/V, \quad f = \Omega_3 h/V \quad (0 \leqslant y \leqslant 1,) \qquad (6.13)$$

then equations $(6.10a$ and $(6.10b)$ become

$$u' = \gamma_1 f+k_1, \quad f''-\gamma_2(f'^2/f)+\gamma_3 f+\gamma_4 = 0, \qquad (6.14a,b)$$

where the prime indicates differentiation with respect to y, and we have eliminated $v_{1,2}$ from $(6.10b)$ and defined new constants:

$$\left.\begin{aligned}\gamma_1 &= \frac{2(2\alpha_6+\alpha_7-\alpha_8)}{2(\alpha_2-\alpha_6)+\alpha_3+\alpha_4-\alpha_7+\alpha_8+\alpha_5/4}, \quad \gamma_2 = \frac{\beta_8}{2\beta_3+\beta_8},\\ \gamma_3 &= \frac{2h^2(2\alpha_6+\alpha_7-\alpha_8)}{2\beta_3+\beta_8}(2+\gamma_1), \quad \gamma_4 = \frac{2k_1h^2(2\alpha_6+\alpha_7-\alpha_8)}{2\beta_3+\beta_8}.\end{aligned}\right\}\qquad (6.15)$$

and k_1 is a constant of integration.

As boundary conditions we take

$$u(0,t) = 0, \quad u(1,t) = 1; \qquad f(0,t) = 0, \quad f(1,t) = 0. \qquad (6.16)$$

The boundary conditions on f, the dimensionless angular velocity, are still subject to investigation. Experimental work in the area of fluid suspensions with preferred directions indicates that conditions (6.16) are physically reasonable (Haynes 1961).

It is possible to make some deductions concerning the physical constants in (6.14). For the isotropic substructure (spherical particles) of Condiff & Dahler (1964), the second term of $(6.14b)$ vanishes. Thus, departure of the substructure particles from the spherical is expressed by the constant γ_2 which is proportional to the particle eccentricity,

$$\gamma_2 = m_1 e, \tag{6.17}$$

where m_1 is a constant and e represents the cross-sectional eccentricity of an oblate or prolate spheroid. Hence, when $e = 0$, the substructure is spherical and equations (6.14) reduce to the same form as those given by Condiff & Dahler for fluids with spherical microstructure.

Both constants γ_3 and γ_4 of (6.14) are proportional to h^2/L^2 where L is some length. If we take L as the largest particle dimension, then as $L \to 0$, i.e. the fluid has no substructure, we obtain from (6.14)

$$\Omega_3 = -\tfrac{1}{2}v_{1.2}, \quad v_{1,2} = m_2,$$

recovering the classical simple shear result that the spin is simply the fluid vorticity and the velocity gradient is a constant.

The constant γ_1 is a measure of the average distance between substructure particles. We arrive at this interpretation by the following: As $\gamma_1 \to 0$, i.e. the distance between particles becomes very small, from (6.15), γ_3 and γ_4 approach zero also. Then the only solution which will satisfy (6.14) and boundary conditions (6.16) is

$$u' = k_1, \quad f = 0,$$

which is physically reasonable since the particles are closely packed when $\gamma_1 \to 0$. Hence γ_1 is related to the concentration of the suspension. Using equation (6.2a) in the form $\gamma_1^{-1}u'' = f'$, it is evident that as γ_1 becomes large, i.e. the distance between particles becomes large, $f' \to 0$, which is, as stated earlier, the condition for negligible interaction of microstructure particles. More exact interpretation of the physical constants requires correlation of experimental data and a complete set of numerical solutions of (6.14) for a range of phenomenological coefficient values. Solution of (6.14) and (6.16) not involving assumptions of small physical constants would require an extensive numerical programme. However, information can be obtained from a geometrical examination of these equations.

6.3. *General geometrical analysis*

We now attempt to analyse the system (6.14), (6.16) from a geometrical point of view. The starting point of this analysis is the first integral of (6.14b) with respect to y, which yields

$$f'^2 = \frac{\gamma_3}{\gamma_2 - 1} f^2 + \frac{2\gamma_4}{2\gamma_2 - 1} f + k_2 f^{2\gamma_2} \quad (\gamma_2 \neq 1, \tfrac{1}{2}), \tag{6.18}$$

where k_2 is a constant of integration. Substituting (6.18) into (6.14b) gives

$$f'' - \frac{\gamma_3}{\gamma_2 - 1} f - k_2 \gamma_2 f^{2\gamma_2 - 1} - \frac{\gamma_4}{2\gamma_2 - 1} = 0. \tag{6.19}$$

If we evaluate (6.18) and (6.19) at $y = 0$ and $y = 1$, six discrete cases become evident and are treated individually below.

Case (1): $\gamma_2 > \frac{1}{2}$, $\gamma_2 \neq 1$

From (6.18) and (6.19),

$$f'(0) = f'(1) = 0, \quad f''(0) = f''(1) = \frac{\gamma_4}{2\gamma_2 - 1}. \tag{6.20}$$

Thus, the function f and its first derivative are zero at both ends of the interval and its second derivative has the same constant value at each end. Then f' must have an odd number of zeros in (0, 1).

Set

$$H(f) = f'^2, \quad a = \frac{\gamma_3}{\gamma_2 - 1}; \quad b = \frac{2\gamma_4}{2\gamma_2 - 1} = 2f''(0) = 2f''(1). \tag{6.21}$$

Then (6.18) becomes

$$H(f) = k_2 f^{2\gamma_2} + af^2 + bf. \tag{6.22}$$

We set $H(f) = 0$ and $f = 0$ is a solution. The reduced equation becomes

$$k_2 f^{2\gamma_2 - 1} + af + b = 0. \tag{6.23}$$

We note that, when $b > 0$, $f'' > 0$ and f must have a positive maximum in (0, 1). When $b < 0$, $f'' < 0$ and f must have a negative minimum in (0, 1).

We assume $H(f)$ continuous and assume it vanishes for some value of f which we denote by f_0. From (6.22), $H(f)$ vanishes for $f = 0$. Now $H'(f)$ is non-vanishing at $f = 0$ and we assume this to be the case at $f = f_0$. Then the integral curves $f(y)$ of (6.18) have well-known properties (see, for example, Synge & Griffith 1959). The most important of these properties for our application are (a) the integral curves are periodic functions between $f = 0$ and $f = f_0$; and (b) they are symmetric with respect to their normals at points of contact with the bounding lines $f = 0$ and $f = f_0$.

If an f_0 does not exist for which $f'(y) = 0$ in (0, 1), then $f(y)$ cannot satisfy the boundary conditions at both ends of the interval. Hence, when a solution exists, it is periodic between $f = 0$ and $f = f_0$ and we limit ourselves to the case of one complete period in the interval [0, 1].

When γ_2 is a positive integer or half integer, it is possible to use the theory of algebraic equations (see, for example, Burnside & Panton 1960) to impose limitations on the signs of the coefficients a, b, k_2. As an example, when γ_2 is a positive integer greater than unity, $2\gamma_2 - 1 = \alpha$, a positive odd integer with $\alpha \geqslant 3$. Then equation (6.23) becomes

$$k_2 f^{\alpha} + af + b = 0. \tag{6.24}$$

Equation (6.24) must have an odd number of zeros since α is odd. But imaginary roots occur in conjugate pairs; hence there will be an odd number of real roots. We can apply Descartes's rule of signs to determine the maximum possible number of real positive and negative roots. It is thus necessary to examine (6.24) for all possible signs of the coefficients.

If k_2, a, b are all greater than zero, there are no positive roots and at most one negative real root. The curve of $H(f)$ as a function of f has the general shape shown in figure 1. Thus $f(y)$ can be periodic between $f = 0$ and $f = -f_0$ (figure 2). But this is impossible because we have taken $b > 0$ and $b = 2f''(0)$ and f must have a positive maximum. Proceeding in this manner, we find that, when γ_2 is a positive integer, only two combinations of coefficient signs are impossible: (1) $k_2 > 0$, $a > 0$, $b > 0$ and (2) $k_2 > 0$, $a > 0$, $b < 0$. When γ_2 is a positive half integer, the following two combinations are impossible: (1) $k_2 > 0$, $a > 0$, $b > 0$ and (2) $k_2 < 0$, $a > 0$, $b < 0$.

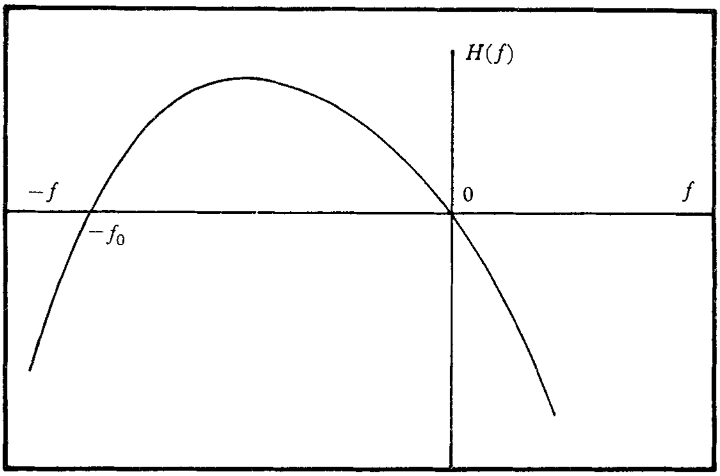

FIGURE 1. The function $H(f)$.

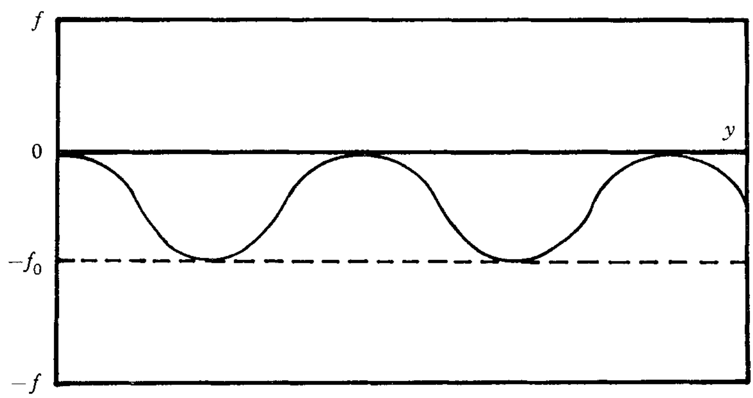

FIGURE 2. The function $f(y)$.

When γ_2 is a positive integer or half integer, it is sometimes possible to determine superior and inferior limits of f_0 in terms of the physical constants. As an example, let $2\gamma_2 - 1$ be an odd positive integer and $k_2 < 0$, $a < 0$, $b > 0$. A solution is possible for this case with $f_0 > 0$. From Burnside & Panton (1960)

$$f_0 < 1 + |b|/[|a| + |k_2|]$$

and $$f_0 > 1/(|a/b| + 1), \quad \text{or} \quad f_0 > 1/(|k_2/b| + 1),$$

whichever is smaller. Since we know nothing about the magnitude of the constant of integration k_2, the best estimates we can get are

$$\left.\begin{array}{l} \dfrac{1}{1+|a/b|} < f_0 < 1 + \dfrac{|b|}{|a|}, \quad k_2 \to 0, \\ 0 < f_0 < 1, \quad k_2 \to \infty. \end{array}\right\} \tag{6.25}$$

Case (2): $0 < \gamma_2 < \frac{1}{2}$

For this case, f remains a periodic function between $f = 0$ and $f = \pm f_0$ when a solution exists. However, no relation exists between $f''(0), f''(1)$ and the physical constants represented by b. There is thus no way of ascertaining whether or not restrictions must be placed on the signs of the coefficients k_2, a, b.

Case (3): $\gamma_2 = 1$

In this case, equation (6.14*b*) becomes

$$f'' - f'^2/f + \gamma_3 f + \gamma_4 = 0. \tag{6.26}$$

The above equation can be integrated once but the result is not subject to geometrical analysis and will require numerical solution. It will not be considered further.

Case (4): $\gamma_2 = \frac{1}{2}$

Equation (6.14*b*) becomes

$$f'' - \tfrac{1}{2} f'^2/f + \gamma_3 f + \gamma_4 = 0. \tag{6.27}$$

The remarks made in Case (3) apply here also.

Case (5): $\gamma_2 = 0$

As stated earlier, $\gamma_2 = 0$ represents the flow of a fluid with isotropic (spherical) microstructure.

Case (6): $\gamma_2 < 0$

From equation (6.18)

$$f'^2(0) = f'^2(1) = \lim_{f \to 0} [k_2/f^{|2\gamma_2|}]. \tag{6.28}$$

Either $k_2 = 0$ or $f'^2(0) = f'^2(1) \to \infty$. We reject the second possibility since it means that the couple stress becomes unbounded at the fluid boundaries, which is not physically plausible.

Then, if $k_2 = 0$, equation (6.18) becomes

$$f'^2 = \frac{\gamma_3}{\gamma_2 - 1} f^2 + \frac{2\gamma_4}{2\gamma_2 - 1} f. \tag{6.29}$$

A geometrical analysis similar to that of case (1) shows that f is periodic between $f = 0$ and $f = f_0$, where

$$f_0 = \frac{-2\gamma_4}{2\gamma_2 - 1} \bigg/ \frac{\gamma_3}{\gamma_2 - 1} = -\frac{b}{a}. \tag{6.30}$$

However, equation (6.29) is directly integrable. The periodic solution is given by

$$y + k_3 = (-a)^{-\frac{1}{2}} \sin^{-1}\left[\frac{-af - b}{b}\right], \tag{6.31}$$

in which $a = \gamma_3/(\gamma_2-1) < 0$. Applying the boundary conditions on f yields

$$(-a)^{\frac{1}{2}} k_3 = \tfrac{3}{2}\pi \pm 2\pi n \quad (n = \text{integer}); \quad (-a)^{\frac{1}{2}} = \pm 2\pi n. \tag{6.32}$$

Thus

$$f = -\frac{b}{a} - \frac{b}{a}\sin\left[\pm 2\pi n(y+1) + \tfrac{3}{2}\pi\right]. \tag{6.33}$$

Note the special form required of the physical constants by equation (6.32).

We can also integrate equation (6.14 a), and, applying the boundary conditions, we obtain

$$u = -\gamma_1\{f_0(-a)^{-\frac{1}{2}} \cos\left[(-a)^{\frac{1}{2}}(y+k_3)\right] + f_0 y\} + k_1 y, \tag{6.34}$$

where $k_1 = 1 + \gamma_1 f_0$ and f_0 is given by (6.30).

6.4. *Special case*, $\gamma_2 = 1\frac{1}{2}$

Equation (6.18) has a solution in familiar form for the case when $\gamma_2 = 1\frac{1}{2}$:

$$f'^2 = k_2 f^2 + 2\gamma_3 f^2 + \gamma_4 f. \tag{6.35}$$

Assume

$$k_2 > 0, \quad \gamma_3 < 0, \quad \gamma_4 > 0. \tag{6.36}$$

Then $f'^2 = 0$ has two positive real roots given by

$$f = -\frac{\gamma_3}{k_2} \pm \frac{\gamma_3}{k_2}\left(1 - \frac{k_2\gamma_4}{\gamma_3^2}\right)^{\frac{1}{2}}, \quad \frac{k_2\gamma_4}{\gamma_3^2} < 1, \tag{6.37}$$

and $f = 0$ is also a root. We arrange the roots so that $r_1 > r_2 > r_3$. Then f is periodic between r_2 and r_3. Let

$$f = r_2\xi^2, \quad \kappa^2 = r_2/r_1, \quad p^* = (k_2 r_1)^{\frac{1}{2}}/2. \tag{6.38}$$

Then

$$\xi' = p^*[(1-\xi^2)(1-\kappa^2\xi^2)]^{\frac{1}{2}}. \tag{6.39}$$

Now let

$$Z = p^* y, \tag{6.40}$$

and (6.39) becomes

$$d\xi/dZ = [(1-\xi^2)(1-\kappa^2\xi^2)]^{\frac{1}{2}}. \tag{6.41}$$

The solution is

$$\xi = \operatorname{sn}(Z + k_4), \tag{6.42}$$

where sn denotes the Jacobian elliptic function. Then, using equations (6.40), (6.38),

$$f = r_2 \operatorname{sn}^2 \tfrac{1}{2}(k_2 r_1)^{\frac{1}{2}} y + k_4), \tag{6.43}$$

where k_4 is a constant of integration. Applying the boundary conditions yields

$$k_4 = 0, \quad \tfrac{1}{2}(k_2 r_1)^{\frac{1}{2}} = 2K, \tag{6.44}$$

for the lowest frequency of $f(y)$, where

$$2K = \int_0^{r_2} \frac{df}{(k_2 f^3 + 2\gamma_3 f^2 + \gamma_4 f)^{\frac{1}{2}}}. \tag{6.45}$$

Making the same change of variables as before,

$$\tfrac{1}{4}k_2 r_1 = F(\tfrac{1}{2}\pi, \kappa), \tag{6.46}$$

where $F(\frac{1}{2}\pi, \kappa)$ is the complete elliptic integral of the first kind. Equation (6.46) determines the constant k_2. For example, when $\gamma_3 = 2{\cdot}0$, $\gamma_4 = 1{\cdot}0$, k_2 is approximately 2·3 and

$$f \simeq 0{\cdot}3 \operatorname{sn}^2 2Ky; \tag{6.47}$$

i.e. f has an amplitude of approximately 0·3.

6.5. *Summary*

In this section we have examined rectilinear shearing motion of fluids with mutually interacting microstructure particles which have a single preferred direction denoted by **n**. This type of fluid can be interpreted as a concentrated fluid suspension.

For the special case when $n_3 = 0$, that is, the particle preferred direction lies in the x_1–x_2 plane, the governing equations are (6.6). These equations are not amenable to analytical methods. However, when they are integrated over a particle period, $T = 2\pi/\Omega_3$, a system of two coupled, non-linear, ordinary differential equations in the linear and spin velocity results. This system, (6.14), is found to contain the effect of average distance between particles (γ_1), particle deviation from spherical (γ_2), and the ratio of some length characteristic of the physical problem to the maximum particle dimension (γ_3 and γ_4). The exact equations for the stresses given by (4.2) and (4.3) exhibit the non-Newtonian phenomena discussed in § 5.

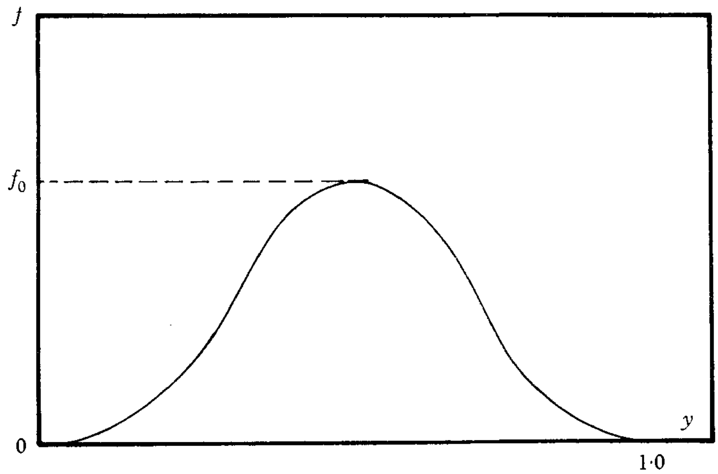

FIGURE 3. The solution $f(y)$ of equation (6.14).

The equations of motion (6.14) are treated from a geometrical standpoint, using as boundary conditions the vanishing of the linear and spin velocity at a fluid boundary. It was found that the first derivative of spin velocity also vanishes at fluid boundaries and that, if a solution exists, it is a periodic function varying between the limits $f = 0$ and $f = f_0$ where $f' = 0$ when $f = f_0$. Moreover, the integral curves $f(y)$ are symmetric about perpendiculars to the bounding lines $f = 0, f = f_0$ at the point of contact (figure 3).

Experimentalists in fluid suspensions have postulated the existence of a 'Magnus force' to explain the axial accumulation of suspended particles and the marginal zone near fluid boundaries which is free of particles. It is our belief that this observed phenomenon is explained from a continuum standpoint by (1) the presence of a pressure gradient in the x_2-direction resulting from the present theory, which can be interpreted as the Magnus force, and (2) the shape of the general dimensionless spin velocity curve in figure 3. This curve, for which $f' = 0$ in the neighbourhood of fluid boundaries, strongly suggests a core of rotating particles and a zone near the walls which is free of particles. Some support for this contention is given by the fact that this axial accumulation effect is not observed for spherical particles. In the present theory, when $n_i = 0$, then $\partial p/\partial x_2 = 0$ and the derivative of the spin velocity does not vanish at fluid boundaries.

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